# The Ergodic Theorem at the Chaotic Pendulum

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**1. Introduction.** Ergodicity is a property associated with chaotic behavior which means that the trajectory of the system in phase space nears arbitrarily close to any energetically possible point. In this paper, I will present my research with a chaotic pendulum on following question: *Do friction affect validity of the Ergodic Theorem and what are the consequences of limited validity?* I do *not* question mathematical validity or ask about its validity range [1-3] but raise the question how the Ergodic Theorem can be applied within its validity range or is influenced by friction.

**2. Methodology.** Evidence-based conclusions can be drawn from data only for the specific system under study. Although significance for related systems seems likely, it cannot be proven methodically. The choice of chaos pendulum (see Fig. 1) as research object is justified in mathematical simplicity [4], but carries a decisive disadvantage: For dimensionality $D$ of the phase space of a system with $n$ degrees of freedom,

$$D = 2n$$

applies [5]. A chaos pendulum has two degrees of freedom; hence its phase space is four-dimensional [6]. Creating four-dimensional phase spaces is a mathematical challenge I was unable to overcome. To solve this problem, I installed a stepper motor that drives the inner pendulum constantly. In this way, feedback between the pendulums is suppressed, whereby the upper angle and associated angular velocity become irrelevant. The pendulum has only one degree of freedom and a two-dimensional phase space anymore. Continuing tension forces are sufficient to allow chaos [7], but have a small enough effect on behavior of the inner pendulum so that its angle and angular velocity can be neglected. For evaluation, I attached a color point to the second pendulum to measure its position using a video camera and tracking program *Viana*. With trigonometry [8], I calculated the angle

$$\tan(\alpha) = \frac{y}{x}$$

and plotted it against angular velocity.

After studying literature [9] I could interpret emerging phase spaces. I varied friction by mounting faces of different sizes in direction of movement [10] and used a mathematic-experimental method to define a friction factor: I deflected the pendulum and inserted high amplitude at the beginning for $A_0$, time for $t$ and lower amplitude after $t$ for $A_t$ in

$$A_t = A_0 \cdot e^{-k*t}.$$

With term rewriting I obtained

$$\frac{A_t}{A_0} = e^{-k \cdot t},$$

by natural logarithm, $e$ was omitted and damping factor $-k$ resulted:

$$\ln\left(\frac{A_t}{A_0}\right) = -k \cdot t.$$

I was now able to vary friction in a reproducible manner.

**3. Measuring Results.** In a one-year measuring series, I observed effects of friction between the pendulums over long periods of time (see Fig. 2). From these diagrams I concluded that friction causes transition to a periodic state, an increase in frequency slows down this process. However, there was a surprise. Apparently, friction affects chaos in two different ways:

i) It leads to transition to a periodic state.

ii) It increases Chaos Entry Frequency (CEF).

The latter was unexpected, but even more: Although damping is nearly proportional to friction, CEF increases only initially quite proportionally, but has a sensitive zone and a limit value (see Fig. 3 and Tab. 1). Because friction slows down chaos, the Ergodic Theorem is *usually not fulfilled*, after all, half-life of chaos is so limited that never every energetically possible point is passed. However, since influence of friction reaches a maximum value, the Ergodic Theorem could be approximately fulfilled at high frequencies.

## 4. Interpretation of the Results.

What are the consequences of these results? To answer this, I dealt with *limit cycles* and interpreted them as attractors [11] which do not pull a system toward a point energy valley, but rather force it into a particular cycle. If it depends on the distance of trajectory´s starting point from limit cycle whether the system diverges or converges, and the distance lies on both x-axis and y-axis and thus represents a volume of energy in phase space, further evolution will show SDIC [12], a new limit-cycle-oriented approach to chaos theory. Indeed, there must be criteria which determine the further evolution. A "catchment area" cannot have a spherical shape, I saw two options for its character:

i) A torus: In this case, *total* energy difference is decisive, the maximum allowed difference would equal torus´ radius.
ii) A fractal: In this case, SDIC would appear because diverging and converging points could exist with an infinitesimal distance to each other.

If there are areas near the limit cycle where it is broken and the Ergodic Theorem is fulfilled at high frequencies (see para. 3), the trajectory will hit an unstable point at t→∞. If it runs exactly back into itself, it is generally unstable because any perturbation grows exponentially [9]. Limit cycles could be unstable. If even phase space itself is fractal, $D$ will then not necessarily have to be a natural, but also a decimal number, which is calculated as *similarity dimension*:

$$D = -\frac{\log(N)}{\log(\varepsilon)},$$

if $N$ is the number of versions of the set itself, reduced by factor ε, of which it consists [13]. This would allow to interpret uncertainty effects, since angle and angular velocity cannot be completely mapped in a less than four-dimensional phase space. However, this is currently a hypothesis and difficult to verify experimentally because uncertainty manifests itself merely by measurement scatter. Furthermore, occurring disturbances rules verification by means of my experimental setup out. I therefore started to design a scheme for a program based on C++.

**5. Creation of a Simulation.** My simulation scheme (see Fig. 4) is applicable to numerous systems. I was faced with the problem to verify my simulation. This is impossible with the chaotic pendulum, because I needed an already verified simulation to solve it at all. I applied it to an already known problem, the Moon´s orbit around Earth (see Fig. 5). I calculated

$$vx_t = vx_0 - G \cdot m_{\oplus} \cdot \frac{x}{r^3 * \Delta t}.$$

On same principle I calculated also $vy_t$. Then $x_t$ is calculated by

$$x_t = x_0 + vx \cdot \Delta t$$

Analogously also $y_t$. Moon´s orbital radius $r$ is valid according to Pythagoras

$$r = \sqrt{x^2 + y^2}.$$

Entered start values correspond to the Moon´s position in its perigee. An optimal value for $\Delta t$ is obtained by variation, I chose $\Delta t = 100\ s$. Similarly, I applied the simulation the Earth´s orbit around Sun - I had just to insert new values:

$$vx_t = vx_0 - G \cdot m_{\odot} \cdot \frac{x}{r^3 \Delta t},$$

Finally, I converted parameters to AU. More difficult is the application to a three-body problem, here nonlinearity occurs which the program needs to be applicable to chaotic systems [14]. I simulated movement of Sun, Earth and Mars under mutual attraction, e. g. for $vx_{\oplus}$:

$$vx_t = vx - G \cdot m_{\odot} \cdot \frac{x}{r_{\odot}^3 \Delta t} - G \cdot m_{♂} \cdot \frac{x}{r_{♂}^3 \Delta t}$$

By factoring out, I obtain:

$$vx_t = vx - \left[ G \frac{x}{\Delta t} \left( \frac{m_{♂}}{r_{♂}^3} + \frac{m_{\odot}}{r_{\odot}^3} \right) \right]$$

Fig. 6 resulted which shows nonlinearities. Afterwards, I put origin on the Earth´s position by coordinate transformation whereby *opposition loops* received (see. Fig.7) [15]. Next step as a circular motion which is simpler than an astronomical three-body problem, but more similar to my chaos pendulum and therefore relevant. In opposite to previous steps, I used vectors and matrices. From rotation angle, I generated a rotation matrix and multiplied it with location vector [16]:

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & -\cos(\alpha) \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

Subsequently, I simulated the second pendulum by setting its origin to $x_t$ and $y_t$ of the first pendulum. Since there are no forces acting yet, feedbacks and thus chaos remain absent (see Fig. 8, 9, 10) [17]. One more important step was introduce friction by inserting friction factor $\mu$ when calculating angular velocity of the second pendulum [18]:

$$\omega_{2_t} = \omega_{2_0} - \mu \cdot \omega_{2_0}.$$

I investigated influence of friction on force-less coupled circular motions and observed that friction slows down movement in a similar way as the chaos, and the two pendulums behave increasingly like one at higher friction. The higher the friction is, the faster the coupled circular motions become one circle, but I was interested in the exact dependence. To work it out, I simulated trajectories with varied $\mu$ and noted the number iterations to a circular motion. The result was a curve with two asymptotes approaching the axes (see Fig. 14): For a friction of 0, it takes an infinite number of iterations; for a friction of 1, it is a single pendulum (see Fig. 11, 12, 13) [19]. I specified my measurements with a regression to a first-degree hyperbola and linearization (see Fig. 15, 16) [20]. As expected, I received an approximate straight line with a surprisingly high determination coefficient $R^2$ of 0. To find out if my program was already subject to SDIC, I developed a method based on Lyapunov exponent $\lambda$ [21] defined by

$$D(t) \approx D_0 2^{\lambda t}.$$

Resolved by $\lambda$ I got

$$\lambda \approx \frac{D(t) - \log(D_0)}{t \cdot \log(2)}.$$

$D(t)$ is the magnitude of difference between both final states after time interval $t$ [22]:

$$D(t) = |x_t - y_t|,$$

Consequently, $D_0$ is defined as

$$D_0 = |x_0 - y_0|.$$

All I need to check my simulation for SDIC is compare initial and final state like this.

**6. Conclusion.** Following results can be verified *for my pendulum*:

i) Chaos is frequency-dependent.

ii) Friction shifts CEF backward.

iii) Friction leads to a transition to periodicity and breaks ergodicity.

iv) At high frequencies, friction loses influence and allows partial ergodicity.

Following is still an open question with at least two possible explanations, which I am currently pursuing (see para. 4):

v) Limit cycles could be unstable. (see 4.a to 5)

It can be concluded on the basis of evidence that Ergodic Theorem does not have unrestricted validity in case of chaotic pendulum and only at high excitation frequencies approaches it. From simulated data, it is not yet possible to clarify type of selection between diverging and converging trajectories. However, the program has demonstrated its agreement with physical reality at various stages of development.