The Ergodic Theorem at the Chaotic Pendulum

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1. Introduction. Ergodicity is a property associated with chaotic behavior which lies in the fact that the trajectory of the system in phase space comes arbitrarily close to any energetically possible point – that´s also the key statement of the Ergodic Theorem. In this paper, I will present my research with a chaotic pendulum on following question: *Do factors of physical reality affect the validity of the Ergodic Theorem and what would be the consequences of limited validity?* I do not question the mathematical validity or ask about its validity range [1-3] but raise the question how the theorem can be applied within its validity range or is influenced by friction – the most relevant physical factor.

2. Methodology. Before I designed an experimental setup, I had started with theoretical considerations based on extensive literature work [4]. I identified several potential influences that may differentiate the physical reality of my real chaotic pendulum from the mathematical model of an idealized one and made a distinction between factors that can be eliminated by modifying the experimental setup and factors that are a fundamental part of our physical reality. I decided to investigate friction as the most fundamental aspect and chose a system that enables friction to be quantified and varied. Evidence-based conclusions can consequently be drawn from the data only for the specific system under study. Although a general significance for related systems seems likely, it cannot be proven methodically.

2.a Dimensionality Problem. The choice of the chaos pendulum as research object was the result of a selection process and is very advantageous, especially because of its mathematical simplicity [5]. However, it also carries one decisive disadvantage: For the dimension D of the phase space of a system with n degrees of freedom the following applies [6]:

$$D = 2n$$

A chaos pendulum has two degrees of freedom; hence its phase space is four-dimensional [7]. However, illustrating four-dimensional phase spaces is a significant mathematic challenge that I was unable to overcome.

2.b Modification of the Setup. To solve this problem, I modified the chaos pendulum. I installed a stepper motor that drives the top pendulum at a constant speed. In this way, the feedback between the pendulums is suppressed, whereby the upper angle and the associated angular velocity become irrelevant.

Therefore, the pendulum has only one degree of freedom and consequently a two-dimensional phase space anymore. The still working tension forces are sufficient to allow feedback and chaotic behavior [8], but have a small enough effect on the behavior of the inner pendulum so that its angle and angular velocity in phase space can be neglected. For evaluation, I attached a colored point to the second pendulum and measured its position using a video camera and the optical evaluation program *Viana*. Using simple trigonometry [9], I was able to calculate the angle from the positions:

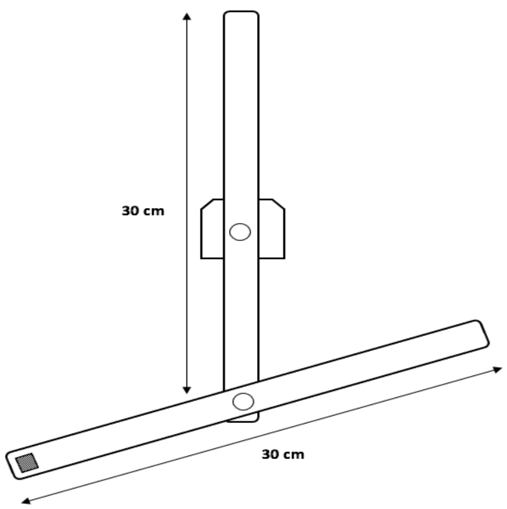

FIGURE 1: Chaos Pendulum Scheme

$$\tan(\alpha) = \frac{y}{x}.$$

I just had to subtract the previous angle from the current and then divide the result by the time step to get an angular velocity [10] which I plotted against the angle and got a phase space [11]. I studied common literature on phase spaces [12] to interpret them correctly.

2.c Variation of friction. Instead of using oils of different viscosity to change the friction, I opted for the simpler method of varying the air resistance by mounting faces of different sizes in the direction of movement of the pendulum [13]. However, it was not certain whether there was a proportionality between the size of the face and the resulting damping. Therefore, I used a mathematical-experimental method to define a universal friction factor. First, I deflected the pendulum and, depending on the area, received a more or less strongly damped sinusoidal oscillation. The damping factor can be derived from the steepness of the line that results when the respective amplitudes are connected [14]. All necessary is inserting the high amplitude at the beginning for A_0, the time for t and the lower amplitude after t for A_t in

$$A_t = A_0 \cdot e^{-k*t}.$$

Term rewriting was first used to divide
by A_0, whereby I obtained

$$\frac{A_t}{A_0} = e^{-k \cdot t},$$

then I calculated the natural logarithm by what e was omitted and I got the damping factor[1]:

$$\ln\left(\frac{A_t}{A_0}\right) = -k \cdot t.$$

I repeated this procedure for faces of different sizes and applied the various damping factors to the surfaces. It was clear: friction is equivalent to damping, there is clearly a proportionality. I was now able to vary friction in a reproducible manner.

3. Measuring Results.

Finally, I started with a one-year series of measurements.

3.a Investigation of friction. Without artificially increasing the friction, I took advantage of the friction between the pendulum rods and observed its effect over long periods of time, on the order of a few days.

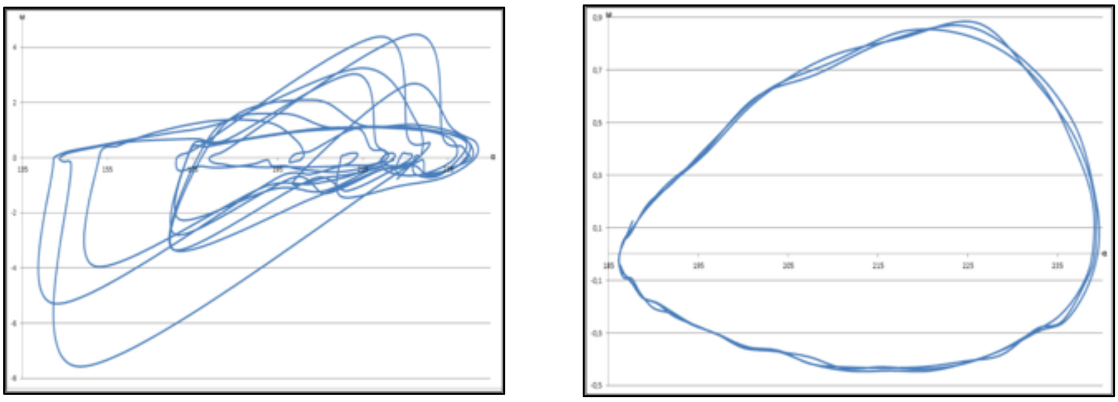

FIGURE 2: First (l.) and last (r.) 100 measuring points plotted in phase space

From these diagrams it can be concluded that friction slows down the chaotic behavior by causing the transition to a periodic state. An artificial increase in friction speeds up

[1] The unit of -k is 1/s.

this process, an increase in frequency slows it down. This result confirmed my conjecture. However, there was also a surprise, because with higher friction, sometimes there was suddenly no more chaos at all. Only when I increased the frequency further did chaos scenario occur again. Apparently, friction affects chaos in two different ways:

i) It leads to a faster transition to a periodic state.

ii) It increases the Chaos Entry Frequency (CEF).

This realization was unexpected, but it got even stranger: Although the damping is nearly proportional to the friction; the CEF is in no way proportional to the damping.

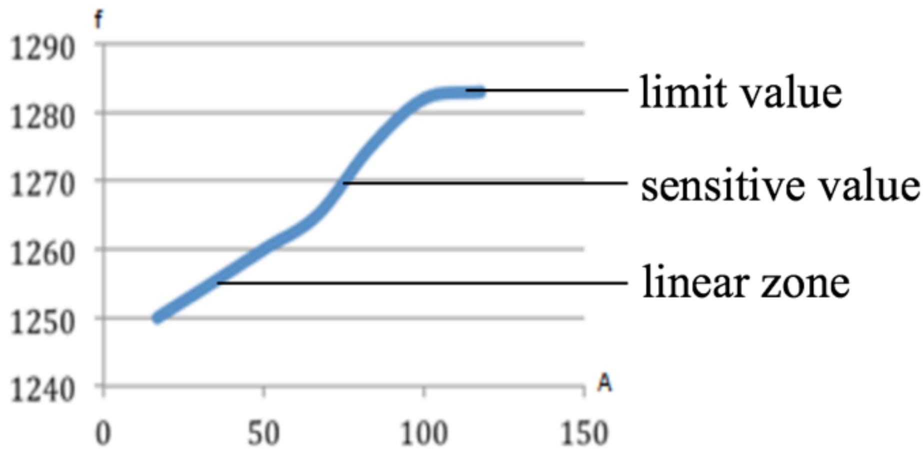

FIGURE 3: CEF plotted against damping factor

It turns out that the friction initially acts quite proportionally, but there is an area in which the friction has a very sensitive effect on the CEF. Then CEF frequency remains on a plateau.

3.b Interpretation of the results. What does this mean for the validity of the Ergodic Theorem? Because friction slows down chaotic behavior, the Ergodic Theorem is usually not fulfilled, after all, the half-life of chaos is so limited that never every energetically possible point is passed. However, since the influence of friction does not grow forever, but at some point, reaches a maximum value, whereas the excitation frequency can be increased further, the Ergodic Theorem is at least approximately fulfilled at high frequencies. Certainly, there is no unrestricted validity, the theorem is a mathematical model with limited applicability on my pendulum.

4. Consequences of the Results. Now I turn to the second part of the question posed at the beginning. Which consequences have the present results?

For this purpose, I delt with *limit cycles.* A limit cycle is an isolated periodic solution of a chaotic system [15], it is characterized by the fact that neighboring trajectories diverge or converge [16]. I described a limit cycle as an attractor in phase space [17] that does not pull a system toward a point-like valley of energy, but rather forces it into a particular cycle that it always strives to break, even when work is expended to break it – the counterpart, so to speak, of chaotic behavior [18]. However, if it depends on the distance of the starting point of the trajectory from the limit cycle whether the system diverges or converges, and the distance lies on both the x-axis and the y-axis and thus represents a volume of energy in phase space, the further behavior of the system would show sensitive dependence on the initial conditions (SDIC) [19]. Thus, I have provided a new limit-cycle oriented approach to classical chaos theory. However, there must be criteria which determine the further evolution. A "zone of attraction" cannot have a spherical shape [20], I saw two options for its character:

i) A torus ("doughnut"): In this case, *total* energy difference is decisive, the maximum allowed difference would equal torus´ radius.
ii) A fractal: In this case, SDIC could appear because diverging and converging points could exist with an infinitesimal distance to each other [21].

However, if there are areas near the limit cycle where the cycle is broken *and* the Ergodic Theorem is fulfilled at high frequencies (see 3b), the trajectory will hit an unstable point at $t \to \infty$ and decay. If it runs exactly back into itself and does not hit an unstable point, it is generally unstable because any perturbation grows exponentially [12]. Limit cycles could inevitable be unstable. On the other hand, fractals could indeed generate "real", i. e. indeterministic, quantum mechanical randomness, since uncertainty would occur according to Heisenberg´s uncertainty principle on extremely small scales – in this case, SDIC would not occur. Maybe even phase space itself is fractal *D* will then not necessarily have to be a natural, but also a

a decimal number, which is calculated as *similarity dimension*:

$$D = -\frac{\log(N)}{\log(\varepsilon)},$$

if N is the number of versions of the set itself, reduced by factor ε, of which it consists [22]. This would allow to interpret uncertainty effects since angle and angular velocity cannot be completely mapped in a less than four-dimensional phase space. However, this is currently a hypothesis and difficult to verify experimentally because uncertainty manifests itself merely by measurement scatter. Furthermore, occurring disturbances rules verification by means of my experimental setup out. I therefore started to design a scheme for a program based on C++.

5. Creation of a Simulation.

It works according to the following principle: For the variables location (x; y) and velocity (vx; vy) initial values are entered as input. From these and the underlying physical laws, a value for the resulting force F and acceleration a is obtained, which is then divided into the components of the velocity change in x- and y-direction (Δvx; Δvy). These can then be used to calculate the new velocities (vx_{n+1}; vy_{n+1}) by simple addition, which is used to determine the new positions (x_{n+1}; y_{n+1}) at the end. These then serve as initial values for the next iteration. After each iteration in the time span Δt, which was also specified at the beginning, the positions should also be plotted, so that I can track the position of the point in real time. This is of elementary importance for the methodology of data collection (see Fig.4).

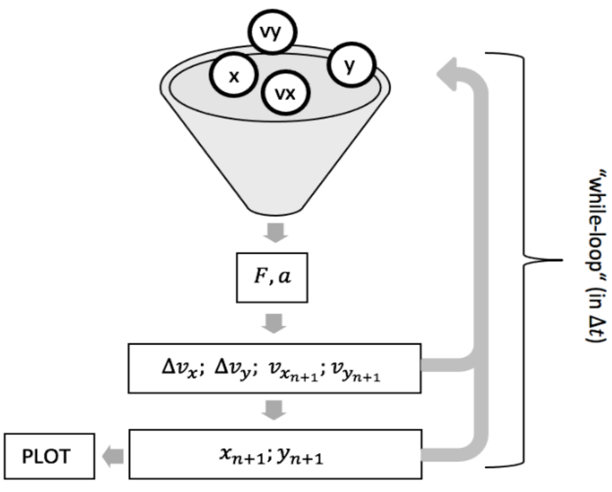

FIGURE 4: Scheme of my Program

5.a Two-Body-Problem. The scheme of my simulation (see Fig.4) can be used to model numerous physical systems. I was faced with the problem that I had to verify my simulation, but this is not possible with my simulated system, because I needed the already verified simulation to be able to solve it at all. Therefore, I started to apply it to an already known problem, the orbit of the moon around earth. For this I needed only a handful of functions, I calculated

$$vx_t = vx_0 - G \cdot m_{\oplus} \cdot \frac{x}{r^3 * \Delta t}.$$

On the same principle I calculated also vy_t, then x_t could be calculated by

$$x_t = x_0 + vx \cdot \Delta t$$

Analogously also y_t. The orbital radius r of the moon is valid according to the Pythagorean theorem

$$r = \sqrt{x^2 + y^2}.$$

The entered start values correspond to the position of the moon in its perigee.[2] An optimal value for Δt can be obtained by variation, I chose $\Delta t = 100\ s$. Then I applied the simulation to another two-body problem, the orbit of the Earth around the Sun. For this I had just to insert new values:

$$vx_t = vx_0 - G \cdot m_{\odot} \cdot \frac{x}{r^3 \Delta t},$$

analogously for vy_t. Then I converted parameters to AU[3] for receiving more manageable numbers and Fig.5 resulted.

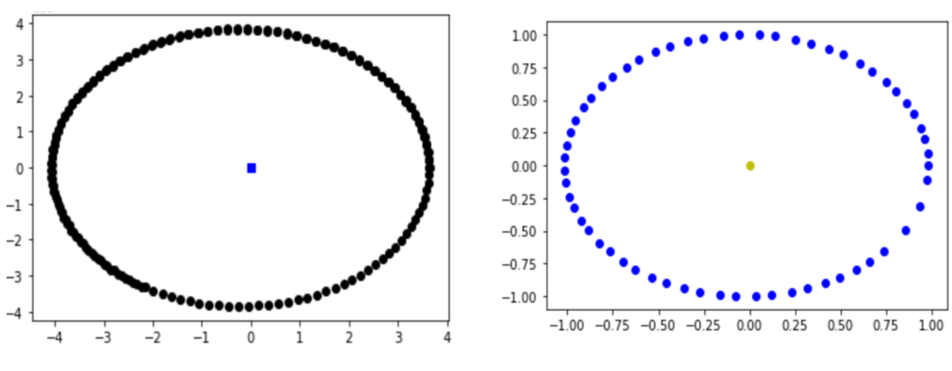

FIGURE 5: Two-Body-Problems Earth-Moon and Sun-Earth plotted in local space

[2] The data was converted to meters by multiplying by 10^3, since G is written in meters.

[3] 1 AU (Astronomical Unit) ≙ 149.597.870.700 m

5.b Three-Body-Problem. Far more difficult is the application to a three-body problem, but here nonlinearity occurs which my program needs to be applicable to my pendulum [24]. I simulated the movement of Sun, Earth and Mars under mutual attraction. The masses and orbital radii of every other body must be taken into account when calculating vx_t and vy_t from the superposition of forces [25], e. g. for $vx_\oplus$:

$$vx_t = vx - G \cdot m_\odot \cdot \frac{x}{r_\odot^3 \Delta t} - G \cdot m_♂ \cdot \frac{x}{r_♂^3 \Delta t}$$

By factoring out, I obtain:

$$vx_t = vx - \left[G \frac{x}{\Delta t} \left(\frac{m_♂}{r_♂^3} + \frac{m_\odot}{r_\odot^3} \right) \right]$$

Same procedure is to be followed with the y-component[4]. Fig.6 resulted which shows nonlinearities for the first time.

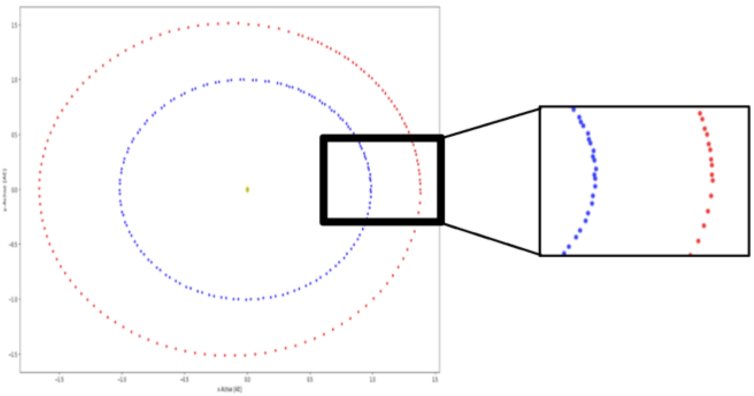

FIGURE 6: Non-linear three-body problem Sun-Earth-Mars plotted in local space

Afterwards, I put the origin on the position of earth by coordinate transformation whereby *opposition loops* received which Mars relative to Earth carries out (see. Fig.7) [26].

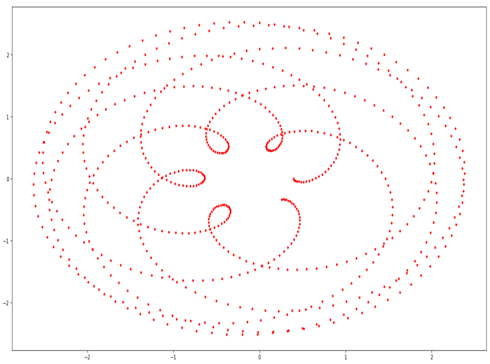

FIGURE 7: Non-linear three-body problem Sun-Earth-Mars with coordinate´s origin on the position of Earth

[4] In this case, r is the distance to earth, not to the origin. Its calculated with $|d_\oplus - d_♂|$, if d is the distance to the origin.

5.c Circular Motion. A circular motion is simpler than an astronomical three-body problem, a "one-body problem" so to speak, but it is also much more similar to my chaos pendulum and therefore relevant. One difference to the previous simulations is that I now worked with vectors and matrices instead of positions and velocities. From the angle of rotation of the pendulum I generated a rotation matrix, which I multiplied with the location vector [27], whereby it changes and generates the new position:

$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & -\cos(\alpha) \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

I checked the result of my simulation by calculating the velocities from the position vectors and plotting their x- and y-components separately. Thereby I got the typical harmonic oscillations, for v_x a sine curve, for v_y a cosine curve [28]. Thus, I could verify the correctness of my software also here several times.

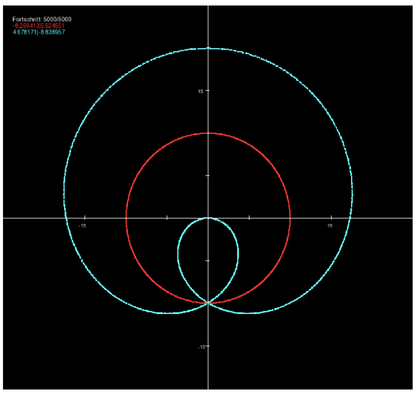

FIGURE 8: Damped circular motions at $\frac{f_2}{f_1} = 1$

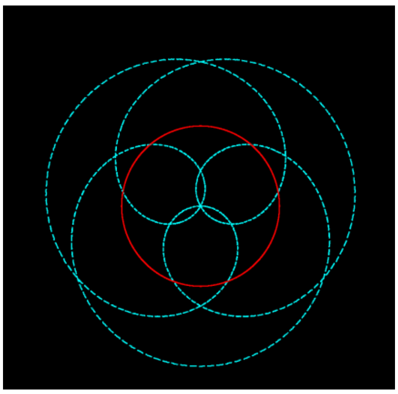

FIGURE 9: Damped circular motions at $\frac{f_2}{f_1} = 4$

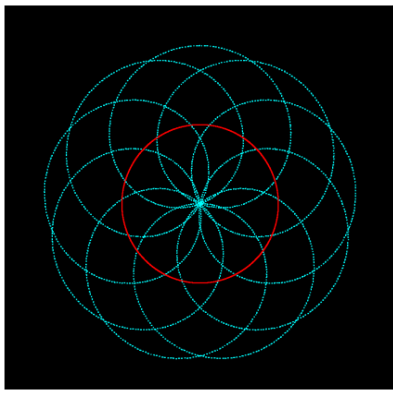

FIGURE 10: Damped circular motions at $\frac{f_2}{f_1} = 10$

5.d Coupled Circular Motion. Subsequently, I simulated also the coupled second pendulum by setting the origin of the second pendulum to x_t and y_t of the first pendulum. This allowed me to generate coupled circular motions – but since there are no forces acting yet, the feedbacks and thus the chaos remain absent (see Fig. 8, 9, 10) [29]. However, regularities could be established, e. g. the number of loops within the path of the first pendulum equals the quotient of the excitation frequencies of the two pendulums minus 1.

5.e Damped Circular Motion. One more important step to simulate my system was introduce friction to be able to determine the influence of friction on chaos later. I inserted factor μ for linear friction when calculating the angular velocity of the second pendulum [30]:

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more important step to simulate my system was introduce friction to be able to determine the influence of friction on chaos later. I inserted factor μ for linear friction when calculating the angular velocity of the second pendulum [30]:

$$\omega_{2_t} = \omega_{2_0} - \mu \cdot \omega_{2_0}$$

I first began to investigate the influence

of friction on force-less coupled circular motions. Thereby it could be observed that the friction slows down the movement in a similar way as the chaos, and the two pendulums behave increasingly like one at higher friction. The higher the friction is, the faster the coupled circular motions become a simple circle.

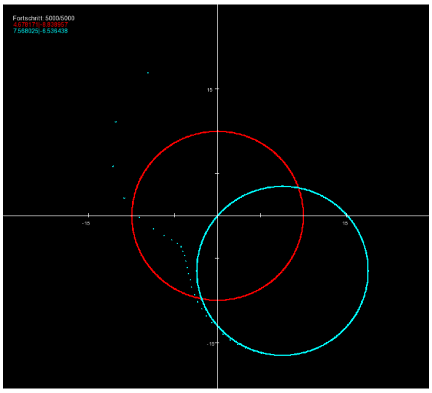

FIGURE 11: Damped circular motions at $\mu = 0.1$

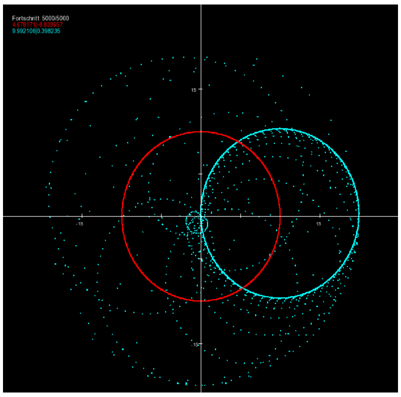

FIGURE 12: Damped circular motions at $\mu = 0.05$

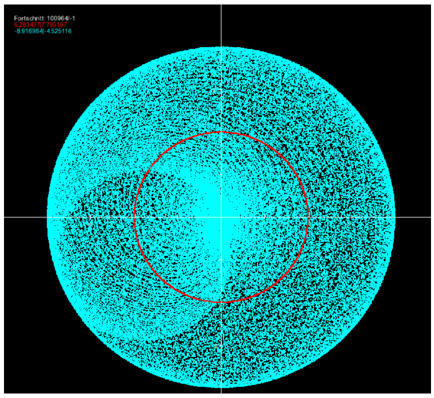

FIGURE 13: Damped circular motions at $\mu = 0.01$

I wondered whether the time until the single circular motion occurs is proportional to the friction and tried to answer this question with my simulation. I simulated trajectories with varied μ and noted after how many iterations a circular motion applied. The result was a curve with two asymptotes approaching the axes: Thus, for a friction of 0, it would take an infinite number of iterations; for a friction of 1, it is a single pendulum (see Fig. 14) [31].

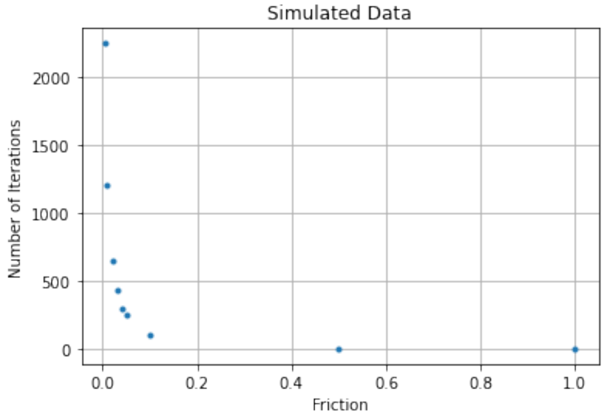

FIGURE 14: Damped circular motions at $\mu = 0.01$

This also is senseful from the scientific perspective. I made my measurements more precise by performing a regression to a first-degree hyperbola and linearizing the curve [32].

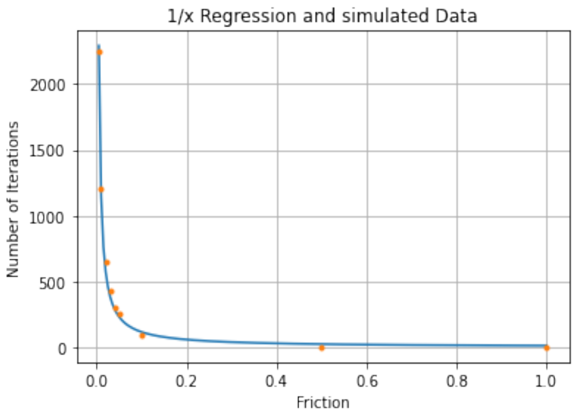

FIGURE 15: Regression of simulated data to a first-degree hyperbola

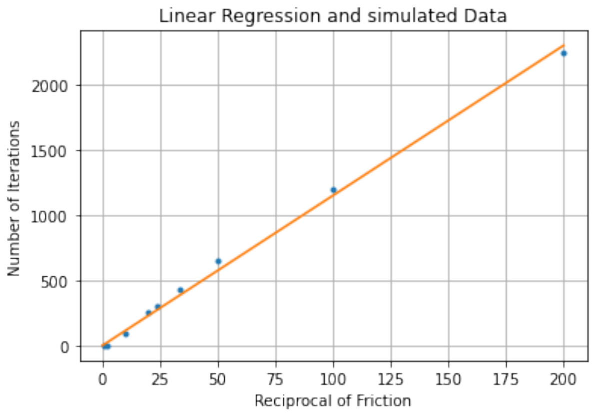

FIGURE 16: Regression of linearized simulated data to a straight

As expected, I received an approximate straight line. The determination coefficient R^2 of 0.997 is surprisingly high for a manual measurement. The antiproportionality factor is 11.4.

5.f Detection of SDIC with Lyapunov exponent. I developed a method based on the Lyapunov exponent to find out if my program was already subject to SDIC. Lyapunov exponent indicates the speed with which two initial states move away from each other within a certain period of time in phase space [33]. One Lyapunov exponent is allotted to each phase space dimension. The exponent λ is defined by

$$D(t) \approx D_0 2^{\lambda t}.$$

Resolved by Lyapunov exponent λ I got

$$\lambda \approx \frac{D(t) - \log(D_0)}{t \ \cdot \log(2)}.$$

$D(t)$ in this case is the magnitude of the difference between the two final states after time period t [34]. Its defined by

$$D(t) = |a_t - b_t|,$$

Consequently, D_0 is defined as

$$D_0 = |a_0 - b_0|.$$

Thus, all I need to do is compare initial and final states in this way, and consider the time interval, to unambiguously check my simulation for SDIC.

6. Conclusion. At this point, I highlight my research results and distinguish between proven and conjectural results. The following results can be verified:

i) At my pendulum, chaos is frequency-dependent. (see 2.c)

ii) At my pendulum, chaos is frequency-dependent. (see 2.c)

iii) Friction shifts the CEF backward. (see 3.b)

iv) Friction leads to a transition to periodicity and breaks ergodicity. (see 3.b)

v) At high frequencies, friction loses its influence and allows partial ergodicity. (see 3.a)

The following, however, is still an open question for which there are at least two different possible explanations, which I
am currently pursuing (see 4.b):

i) Limit cycles could be unstable or the phase space has a fractal dimension. (see 4.a to 5)

The research question posed at the beginning can be answered comprehensively with these results: In the case of my chaos pendulum, the ergodic theorem is not unrestrictedly valid. At high frequencies it can potentially have approximate validity. However, the question raised in 4.a cannot yet be answered from the available data. It is not possible to infer the nature of the selection between diverging and converging trajectories, but the program in development has demonstrated its agreement with physical reality at various stages of development.

SUPPLEMENTARY MATERIAL

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