Chaos Theory: Is the Ergodic Theorem valid in reality?

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Introduction

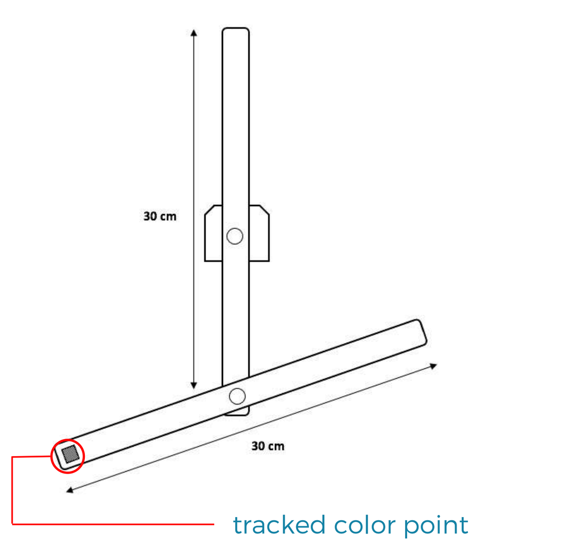
My research question concerns the so-called Ergodic Theorem and its range of validity. The Ergodic Theorem, formulated by Ludwig Boltzmann in a slightly different version in 1887, describes one typical aspect of chaotic systems, the Ergodicity – a property of a system which means that the trajectory comes arbitrarily close to any energetically possible point in phase space (molecular chaos). The Ergodic Theorem also makes statements about the time after which a point is passed, but this aspect is irrelevant for my research. I posed the following research question: **What is the scope of application of the Ergodic Theorem at the chaotic pendulum and what are the consequences of a limited validity?** However, this doesn´t mean to which systems the Ergodic Theorem is applicable and which exceptions exist - this is already quite clear (s. research about spontaneous symmetry breaking by Nambu, Kobayashi and Maskawa). Instead, I want to find out to what extent the theorem formulated as a mathematical model can actually be applied to a system, which, at least theoretically, disprove the theorem: the chaotic pendulum. I am interested in factors not taken into account in the model and whether they impair its impair its applicability in case of the chaotic pendulum.

Methodology

model	reality
frictionless	undefined period friction between my pendulums
no air resistance	air resistance at the pendulum´s rods
six-dimensional	three-dimensional
undefined period	period max. 2 days
infinite accuracy	optical evaluation
no centrifugal force	centrifugal force increasing with frequency
exactly constant environmental conditions	small deviations

Before I designed my final experimental setup, I had started with theoretical considerations based on extensive literature work. I identified several potential influences that may differentiate the physical reality of my real chaotic pendulum from the mathematical model of an idealized one.

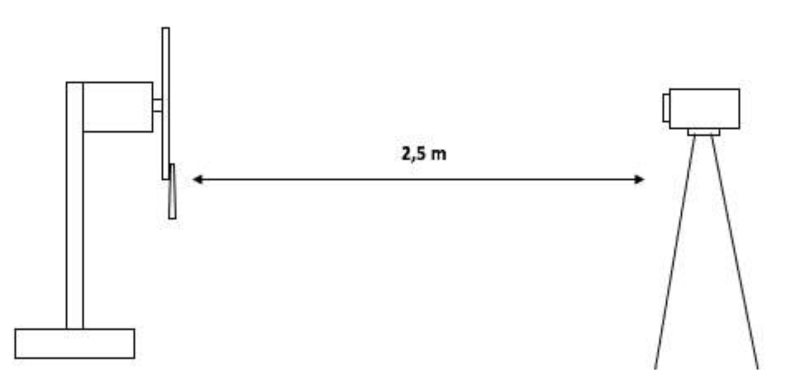
To answer the question, I made a distinction between factors that can be eliminated by modifying the setup and factors which are a fundamental part of our physical reality. I decided to investigate friction as the most fundamental aspect with my chaotic pendulum.



To build a chaos pendulum, only two pendulums rods have to be coupled together, so that the second pendulum can behave chaotically. However, this choice also has some disadvantages: Although the centrifugal force turned out to be negligible at the frequencies examined, there is a problem with the representation and evaluation. For the dimension D of the phase space of a system with n degrees of freedom the following applies:

$$D = 2\text{n}$$

A chaotic pendulum has two degrees of freedom; hence its phase space is four-dimensional. It takes four values to describe the state of a chaos pendulum. However, creating four dimensional phase spaces is a mathematic challenge which I wasn´t able to get along with.

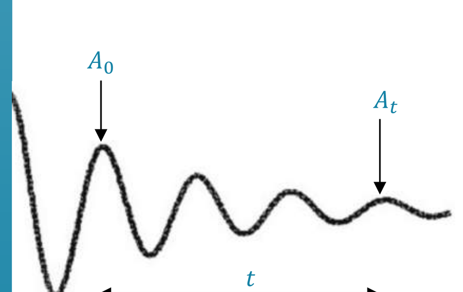


To solve this problem, I modified the pendulum by installing a stepper motor which drives the top pendulum at a constant speed. On this way, the feedback between stages is suppressed, whereby the upper angle and the associated angular velocity become irrelevant. Thus, the pendulum has one degree of freedom and hence a two-dimensional phase space anymore.

I developed a semi-optic evaluation process. I attached a colored point to the second pendulum and measured its position using a precise video camera and an evaluation program. Using simple trigonometry, I was able to calculate the angle from the positions:

$$\tan(\alpha) = \frac{y}{x}$$

I just had to subtract the previous angle form the current and then divide the result by the time step to get an angular velocity which I then plotted against the angle and got phase spaces. First, I interpreted the system´s trajectory in the phase space optically with help of extensive literature.



Then, I developed a way to quantify and vary the influence of friction. Instead of using oils of different viscosity to change the friction, I opted for the simpler method of varying air resistance by mounting faces of different size in direction of movement. However, it wasn´t certain if there´s a proportionality between area and damping. Thus, I used a mathematical method to define an damping factor: I deflected the pendulum and, depending on area, received a more or less strongly damped oscillation. I used the equation

$$A_t = A_0 \cdot e^{-k \cdot t}$$

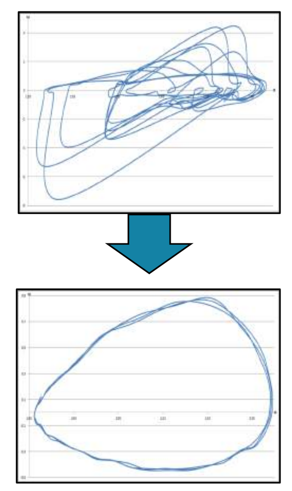
and obtained by term transformation first

$$\frac{A_t}{A_0} = e^{-k \cdot t}$$

and then

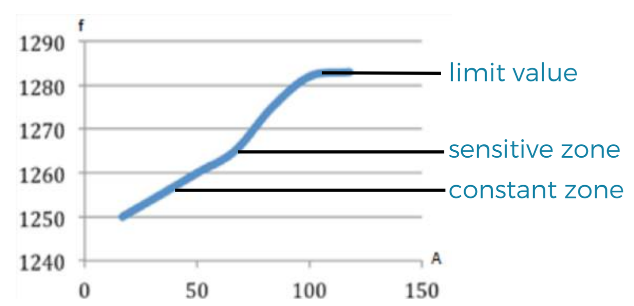
$$\ln\left(\frac{A_t}{A_0}\right) = -k \cdot t.$$

My Research Results



I started with a simple measurement without increasing the friction further, I used friction between the pendulums and observed its long-term effect over a few days. In the first you can see the first 100 measuring points with rollovers of the lower pendulum in irregular distances. On this way, different periods emerge who overlay each other.

In the picture below, the last 100 measuring points are plotted. Although nonlinearities are present, the pendulum does not perform a rollover. This quasi-periodic state does normally not change anymore, small disturbances cannot change it in the long term. This was no surprising discovery.



However, there was a surprise elsewhere: With higher friction, sometimes there was suddenly no more chaos at all. Apparently, friction affects chaos in two different ways.

i. It leads to a faster transition to a periodic state.
ii. It shifts the chaos entry frequency backwards.

Damping Factor	Chaos Entry Frequency
0.145/s	4.6370 rad/s
0.178/s	4.6496 rad/s
0.204/s	4.6684 rad/s
0.305/s	4.6873 rad/s
0.497/s	4.7250 rad/s
0.600/s	4.7501 rad/s

This realization was most unexpected, but it got even stranger. Although the damping is completely proportional to the friction the damping is not proportional to the chaos entry frequency. There´s an rea in which friction has a very sensitive effect on the chaos entry frequency. Then chaos entry frequency remains on a plateau.

Possible Consequences

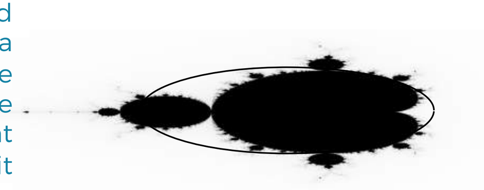
What does this mean for the the Ergodic Theorem? Because friction slows down chaos, the Theorem is usually not fulfilled at my pendulum - the half-life of chaos is so limited that never every energetically possible point is passed. However, since the influence of friction doesn´t grow forever, but at some point reaches a maximum value, whereas the excitation frequency can be increased further, the Ergodic Theorem is at least approximately fulfilled at high frequencies. Certainly, **there is no unrestricted validity**.

This could have interesting consequences for limit cycles, quasi-periodic isolated solutions of nonlinear equations. A limit cycle is a attractor that forces a system into a cycle instead of a region. If it depends on the distance of the starting point to the limit cycle whether the system diverges or converges, and the distance lies axes and represents a volume of energy in phase space, the further behavior of the system will carry SDIC. Thus, I have provided a new limit-cycle oriented approach to the classical chaos theory.

However, if the course of the system depends on the distance to the cycle, i.e. on the energy difference, then the questions emerges how the selection between diverging and converging works in detail. I postulated three different options, current data doesn´t prefer one of them.

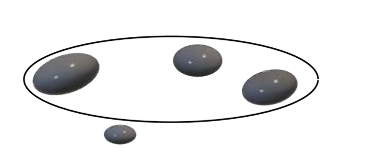
1. Random (well, not really):

A selection by chance could "hid" behind the postulated SDIC and would be difficult to differ from it. Of course, a fractal cannot produce real random, because of the infinitesimal distances it´s just very difficult to model the process. Consequently, after $t \to \infty$ an unstable point would be hit and the cycle would decay. Otherwise, it would run back into itself and also decay.



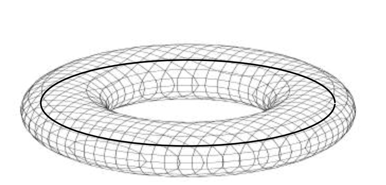
2. Initial Energy:

In this case, different energy spans would converge and others would diverge. At the edge of this areas, there would be a SDIC – theoretically, the cycle could run over long time periods without hitting one of these regions and show SDIC.

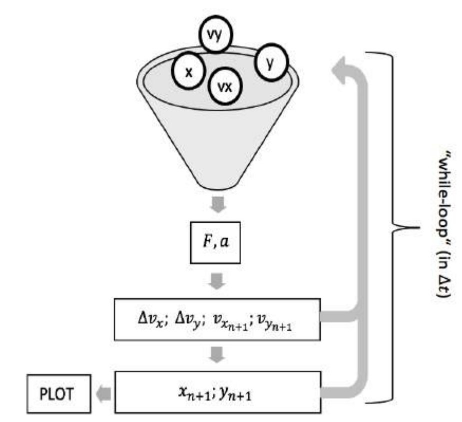


3. Total Energy Difference:

This case would be the only stable scenario. All starting points near the limit cycle would lead to a converging cycle, the radius of the torus would correspond to the maximum energy difference of a converging trajectory. If two initial values, which differ only by an infinitesimal amount, don´t always **both** diverge or converge, this option can be rejected. I decided to create a simulation.



Development of a Simulation



The scheme of my simulation can be used to model numerous physical systems. I was faced with the problem that I had to verify my simulation, but this isn´t possible with the simulated system because I need the already verified simulation to be able to solve it at all. Therefore, I started to apply it to an already known problem, the orbit of the moon around earth. For this I needed only a handful of functions, I calculated

$$vx_t = vx_0 - G \cdot m_{\oplus} \cdot \frac{x}{r^3 \Delta t}$$

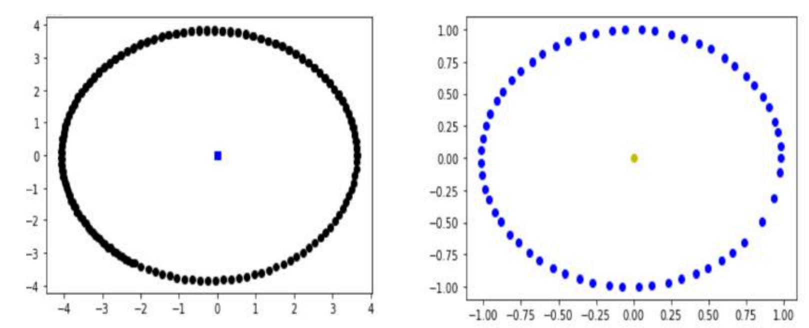
With the same principle I calculated also the y-component vy_t, then x_t could be calculated by

$$x_t = x_0 + vx \cdot \Delta t$$

Analogously also y_t. The orbital radius r of the moon is valid according to Pythagorean theorem:

$$r = \sqrt{x^2 + y^2}$$

The entered start values correspond to the position of the moon in its perigee. An optimal value for Δt can be obtained by variation, I chose $\Delta t = 100\ s$.



Then I applied the simulation to another two-body problem, the orbit of the earth around the sun. For doing this, I just had to insert new values:

$$vx_t = vx_0 - G \cdot m_{\odot} \cdot \frac{x}{r^3 \Delta t}.$$

Analogously for vy_t. Then I converted parameters to AU.

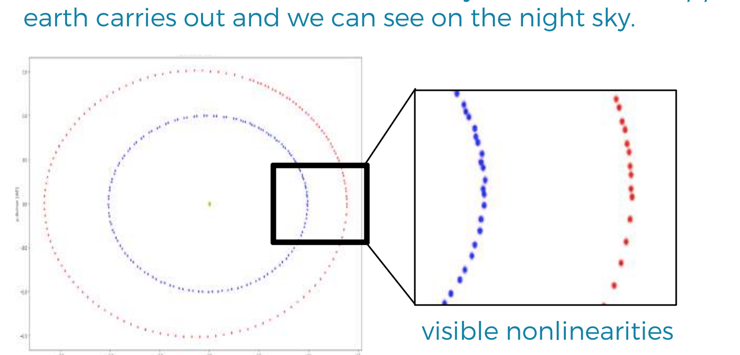
Far more difficult is the application to a three-body problem, but here nonlinearity occurs which my program needs to be applicable to my pendulum. I simulated the movement of Sun, Earth and Mars under mutual attraction. The masses and orbital radii of every other body must be taken into account when calculating vx_t and vy_t from the superposition of all forces, e. g. for $vx_{\oplus}$:

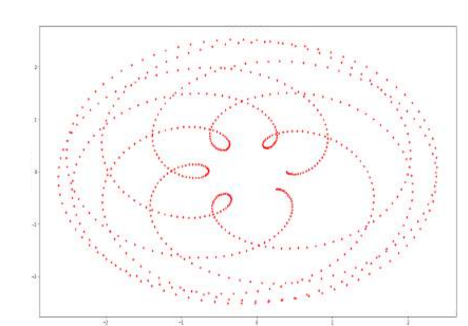
$$vx_t = vx - G \cdot m_{\odot} \cdot \frac{x}{r_{\odot}^3 \Delta t} - G \cdot m_{♂} \cdot \frac{x}{r_{♂}^3 \Delta t}$$

By factoring out, I obtained

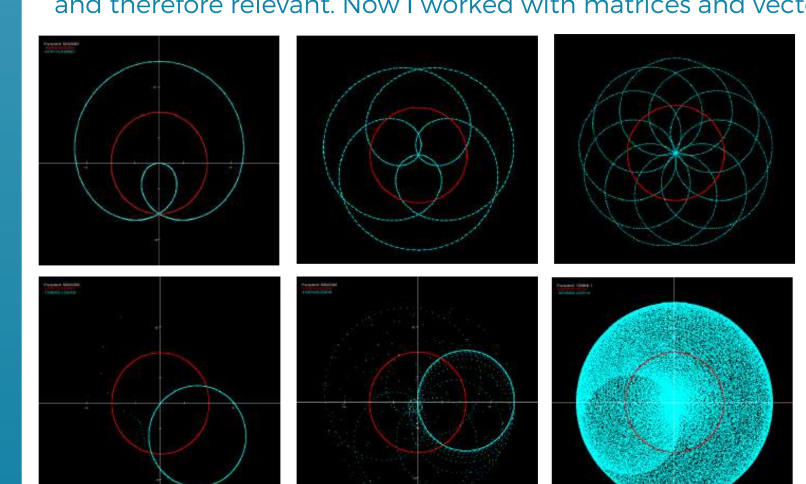
$$vx_t = vx - \left[G \frac{x}{\Delta t} \left(\frac{m_{♂}}{r_{♂}^3} + \frac{m_{\odot}}{r_{\odot}^3} \right) \right]$$

After I repeated the same procedure with all other calculated values, I received the first figure with visible nonlinearities. Afterwards I put the origin on the position of the earth by means of coordinate transformation, whereby I received the *opposition loops* which Mars relative to the earth carries out and we can see on the night sky.





A circular motion is simpler than a three-body-problem, but also more similar to my pendulum and therefore relevant. Now I worked with matrices and vectors instead of positions and velocities.



From the angle of rotation of the pendulum I generated a rotation matrix which I multiplied with the location vector.

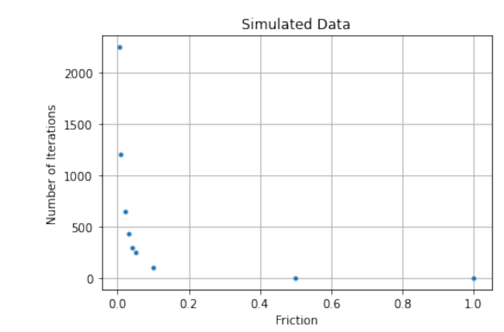
$$\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & -\cos(\alpha) \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

To reconstruct my pendulum, I just set the origin of the second pendulum to the position of the first. Then I introduced linear friction by inserting the factor μ into

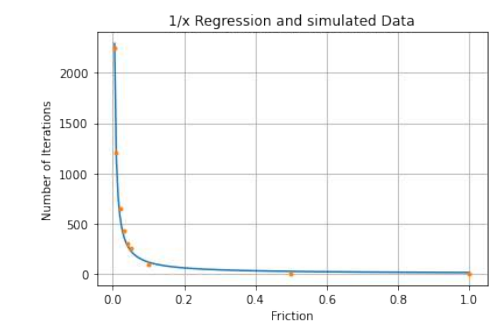
$$\omega_{2_t} = \omega_{2_0} - \mu \cdot \omega_{2_0}.$$

Further Research with the Simulation

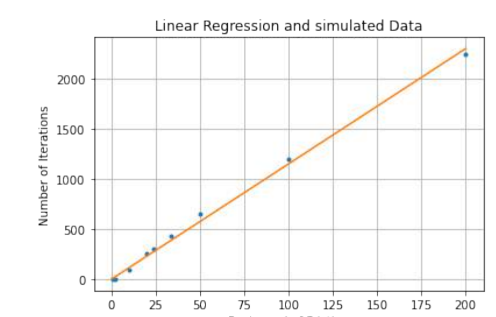
Afterwards, I began to applicate my simulation on some physical questions. I wanted to investigate the dependence of the number of iterations after which the quasi-periodic state is reached on the friction. I varied friction and counted the iterations to periodicity.



These were the original data, I concentrated my measurements on the critical area with high ascent. At first, the curve gave the impression of a hyperbola. This would be senseful, because there is no chaos at all (zero iterations to periodicity) with a friction of one and chaos forever (infinite number of iterations) with no friction.



To proof or reject this assumption, I ran a mathematical regression. I wanted to find out not only if it was a hyperbola, but also what degree hyperbola it was. First I assumed a hyperbola of first degree and got an impressive conformity of 0.997. The infinitesimal difference is due to the fact that I stopped the simulation manually when periodicity occurred.



For further confirmation, I linearized the hyperbola by plotting the reciprocal of the friction on the x-axis. Here, as expected, I obtained a nearly straight line. This clearly proves that the number of iterations to periodicity is inversely proportional to friction with an antiproportionality factor of 11.4

These solid results prove the suitability of my program. In the next step, I built coupled differential equations into the simulation to create a feedback between the two pendulums and thus allow for final chaos. The current results give the impression of chaotic behavior. To what extent there is a SDIC, I will determine by finding out the Lyapunov exponents of the respective phase space dimensions.

Conclusion

The following conclusions can be drawn from my research findings:

- The chaotic behavior of the driven chaos pendulum is frequency-dependent. ✓ **proven**
- Internal and external friction shift the chaos entry frequency to higher frequencies **and** lead to a faster transition to a periodic state. ✓ **proven**
- At high frequencies, the friction loses influence on my chaotic pendulum ✓ **proven**
- The Ergodic Theorem is only partially fulfilled for my chaotic pendulum. ✓ **proven**
- The limit cycle of my chaos pendulum could not be stable. ? **supposed**

The following questions must be answered to solve the final puzzle:

- Is selection between diverging and converging trajectories subject to chance or distance from the limit cycle for semi-stable limit cycles?
- Is there an equivalence between the shift in chaos entry frequency and the longer duration to reach periodicity?
- Does the catchment area perhaps take the form of a torus surrounding the limit cycle?

The project title and conclusion may suggest that the general validity of the ergodic theorem is in question or that its practical applicability is limited. However, this is neither done nor refuted by my research, certainly the results can only be applied to the chaos pendulum. To what extent other systems are affected may be the subject of future research.